

# RIGHT $n$ -ANGULATED CATEGORIES ARISING FROM COVARIANTLY FINITE SUBCATEGORIES

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**ABSTRACT.** We define the notion of right  $n$ -angulated category, which generalizes the notion of right triangulated category. Let  $\mathcal{C}$  be an additive category or  $n$ -angulated category and  $\mathcal{X}$  a covariantly finite subcategory, we show that under certain conditions the quotient  $\mathcal{C}/\mathcal{X}$  is a right  $n$ -angulated category. This result generalizes some previous work.

## 1. INTRODUCTION

Geiss, Keller and Oppermann introduced the notion of  $n$ -angulated category, which is a “higher dimensional” analogue of triangulated category [6]. Bergh and Thaule introduced a higher “octahedral axiom” for an  $n$ -angulated category and showed that it is equivalent to the mapping cone axiom [3]. Certain  $(n-2)$ -cluster tilting subcategories of triangulated categories [6] and finitely generated free modules over certain local algebras [4] give rise to  $n$ -angulated categories. Recently, Jasso introduced  $n$ -abelian categories,  $n$ -exact categories and algebraic  $n$ -angulated category [8]. He showed that the quotient category of a Frobenius  $n$ -exact category has a natural structure of  $(n+2)$ -angulated category, which generalizes Happel’s Theorem [7, Theorem 2.6].  $N$ -angulated quotient categories induced by mutation pairs were discussed in [9]. Other properties of  $n$ -angulated categories can see [5].

Beligiannis and Marmaridis defined the notion of right triangulated category and showed that if  $\mathcal{X}$  is a covariantly finite subcategory of an additive category  $\mathcal{C}$ , and if any  $\mathcal{X}$ -monic has a cokernel, then the quotient  $\mathcal{C}/\mathcal{X}$  has a structure of right triangulated category. The main aim of this paper is to define the notion of right  $n$ -angulated category, and discuss the right  $n$ -angulated categories arising from covariantly finite subcategories. Our two main results, see Theorem 3.5 and Theorem 3.8 for details, will generalize some previous work such as [2, Theorem 2.12], [8, Theorem 5.11], [9, Theorem 3.8] and [1, Theorem 7.2].

This paper is organized as follows. In Section 2, we define the notion of right  $n$ -angulated category, recall the definitions of covariantly finite subcategory,  $n$ -cokernel and  $n$ -pushout, and give some preliminaries. In Section 3, we state and prove our main results, then give some applications.

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## 2. DEFINITIONS AND PRELIMINARIES

In this section we define the notion of right  $n$ -angulated category, then give some other preliminaries.

Let  $\mathcal{C}$  be an additive category equipped with an endofunctor  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  and  $n$  an integer greater than or equal to three. An  $n$ - $\Sigma$ -sequence in  $\mathcal{C}$  is a sequence of morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1.$$

Its *left rotation* is the  $n$ - $\Sigma$ -sequence

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2.$$

A *morphism of  $n$ - $\Sigma$ -sequences* is a sequence of morphisms  $\varphi = (\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n)$  such that the following diagram commutes

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

where each row is an  $n$ - $\Sigma$ -sequence. It is an *isomorphism* if  $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$  are all isomorphisms in  $\mathcal{C}$ .

**Definition 2.1.** A *right  $n$ -angulated category* is a triple  $(\mathcal{C}, \Sigma, \Theta)$ , where  $\mathcal{C}$  is an additive category,  $\Sigma$  is an endofunctor of  $\mathcal{C}$ , and  $\Theta$  is a class of  $n$ - $\Sigma$ -sequences (whose elements are called right  $n$ -angles), which satisfies the following axioms:

(RN1) (a) The class  $\Theta$  is closed under isomorphisms, direct sums and direct summands.

(b) For each object  $X \in \mathcal{C}$  the trivial sequence

$$X \xrightarrow{1_X} X \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma X$$

belongs to  $\Theta$ .

(c) For each morphism  $f_1 : X_1 \rightarrow X_2$  in  $\mathcal{C}$ , there exists a right  $n$ -angle whose first morphism is  $f_1$ .

(RN2) If an  $n$ - $\Sigma$ -sequence belongs to  $\Theta$ , then its left rotation belongs to  $\Theta$ .

(RN3) Each commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow \varphi_n & & \downarrow \Sigma \varphi_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in  $\Theta$  can be completed to a morphism of  $n$ - $\Sigma$ -sequences.

(RN4) Given a commutative diagram

$$\begin{array}{ccccccccccc}
 X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
 \parallel & & \downarrow \varphi_2 & & & & & & & & & & \parallel \\
 X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\
 & & \downarrow h_2 & & & & & & & & & & \\
 & & Z_3 & & & & & & & & & & \\
 & & \downarrow h_3 & & & & & & & & & & \\
 & & \vdots & & & & & & & & & & \\
 & & \downarrow h_{n-2} & & & & & & & & & & \\
 & & Z_{n-1} & & & & & & & & & & \\
 & & \downarrow h_{n-1} & & & & & & & & & & \\
 & & Z_n & & & & & & & & & & \\
 & & \downarrow h_n & & & & & & & & & & \\
 & & \Sigma X_2 & & & & & & & & & & 
 \end{array}$$

whose top rows and second column belong to  $\Theta$ . Then there exist morphisms  $\varphi_i : X_i \rightarrow Y_i (i = 3, 4, \dots, n)$ ,  $\psi_j : Y_j \rightarrow Z_j (j = 3, 4, \dots, n)$  and  $\phi_k : X_k \rightarrow Z_{k-1} (k = 4, 5, \dots, n)$  with the following two properties:

- (a) The sequence  $(1_{X_1}, \varphi_2, \varphi_3, \dots, \varphi_n)$  is a morphism of  $n$ - $\Sigma$ -sequences.
- (b) The  $n$ - $\Sigma$ -sequence

$$\begin{aligned}
 X_3 & \xrightarrow{\begin{pmatrix} f_3 \\ \varphi_3 \end{pmatrix}} X_4 \oplus Y_3 \xrightarrow{\begin{pmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \phi_4 & \psi_3 \end{pmatrix}} X_5 \oplus Y_4 \oplus Z_3 \xrightarrow{\begin{pmatrix} -f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \phi_5 & \psi_4 & h_3 \end{pmatrix}} X_6 \oplus Y_5 \oplus Z_4 \\
 & \xrightarrow{\begin{pmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \phi_6 & \psi_5 & h_4 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n-1}\varphi_{n-1} & -g_{n-2} & 0 \\ \phi_{n-1} & \psi_{n-2} & h_{n-3} \end{pmatrix}} X_n \oplus Y_{n-1} \oplus Z_{n-2} \\
 & \xrightarrow{\begin{pmatrix} (-1)^n \varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{pmatrix}} Y_n \oplus Z_{n-1} \xrightarrow{(\psi_n, h_{n-1})} Z_n \xrightarrow{\Sigma f_2 \cdot h_n} \Sigma X_3
 \end{aligned}$$

belongs to  $\Theta$ , and  $h_n \cdot \psi_n = \Sigma f_1 \cdot g_n$ .

**Remarks 2.2.** (a) If  $n = 3$ , then the right  $n$ -angulated category  $(\mathcal{C}, \Sigma, \Theta)$  is a right triangulated category defined by Beligiannis and Marmaridis [2].

(b) If  $\Sigma$  is an equivalence, and the condition in axiom (RN2) is necessary and sufficient, then the right  $n$ -angulated category  $(\mathcal{C}, \Sigma, \Theta)$  is an  $n$ -angulated category in the sense of Geiss-Keller-Oppermann. See [6, Definition 1.1] and [3, Theorem 4.4].

(c) We can define left  $n$ -angulated category dually. If  $(\mathcal{C}, \Sigma, \Theta)$  is a right  $n$ -angulated category,  $(\mathcal{C}, \Omega, \Phi)$  is a left  $n$ -angulated category,  $\Omega$  is a quasi-inverse of  $\Sigma$  and  $\Theta = \Phi$ , then  $(\mathcal{C}, \Sigma, \Theta)$  is an  $n$ -angulated category.

Let  $\mathcal{C}$  be an additive category and  $\mathcal{X}$  a subcategory of  $\mathcal{C}$ . Throughout this paper, when we say that  $\mathcal{X}$  is a subcategory of  $\mathcal{C}$ , we always mean that  $\mathcal{X}$  is full and is closed under isomorphisms, direct sums and direct summands. We recall that the quotient category  $\mathcal{C}/\mathcal{X}$  has the same objects as  $\mathcal{C}$  and that the set of morphisms  $(\mathcal{C}/\mathcal{X})(A, B)$  is defined as the quotient group  $\mathcal{C}(A, B)/[\mathcal{X}](A, B)$ , where  $[\mathcal{X}](A, B)$  is the set of morphisms from  $A$  to  $B$  which factor through some object of  $\mathcal{X}$ . For any morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , we denote by  $\underline{f}$  the image of  $f$  under the quotient functor  $\mathcal{C} \rightarrow \mathcal{C}/\mathcal{X}$ .

A morphism  $f : A \rightarrow B$  in  $\mathcal{C}$  is called  $\mathcal{X}$ -*monic* if  $\mathcal{C}(B, X) \xrightarrow{\mathcal{C}(f, X)} \mathcal{C}(A, X) \rightarrow 0$  is exact for any object  $X \in \mathcal{X}$ . A morphism  $f : A \rightarrow X$  in  $\mathcal{C}$  is called a *left  $\mathcal{X}$ -approximation of  $A$*  if  $f$  is  $\mathcal{X}$ -monic and  $X \in \mathcal{X}$ . The subcategory  $\mathcal{X}$  is said to be a *covariantly finite subcategory* of  $\mathcal{C}$  if any object  $A$  of  $\mathcal{C}$  has a left  $\mathcal{X}$ -approximation. We can define  $\mathcal{X}$ -*epic* morphism, *right  $\mathcal{X}$ -approximation* and *contravariantly finite subcategory* dually. The subcategory  $\mathcal{X}$  is called *functorially finite* if it is both contravariantly finite and covariantly finite.

Let  $f : A \rightarrow B$  be a morphism in  $\mathcal{C}$ . A *weak cokernel* of  $f$  is a morphism  $g : B \rightarrow C$  such that  $gf = 0$  and for each morphism  $h : B \rightarrow D$  with  $hf = 0$  there exists a (not necessarily unique) morphism  $k : C \rightarrow D$  such that  $h = kg$ . It is easy to see that a weak cokernel  $g$  of  $f$  is a cokernel of  $f$  if and only if  $g$  is an epimorphism.

**Lemma 2.3.** ([8, Lemma 2.1]) *Let  $A = (A_i, d_i)$  and  $B = (B_i, d'_i)$  be complexes concentrated in nonnegative degrees, such that for all  $i \geq 0$  the morphism  $d_{i+1}$  is a weak cokernel of  $d_i$ . If  $f : A \rightarrow B$  and  $g : A \rightarrow B$  are morphisms of complexes such that  $f_0 = g_0$ , then there exists a homotopy  $h$  between  $f$  and  $g$  such that  $h_0 = 0$ .*

**Definition 2.4.** ([8, Definition 2.2]) Let  $\mathcal{C}$  be an additive category and  $f_0 : A_0 \rightarrow A_1$  a morphism in  $\mathcal{C}$ . An  *$n$ -cokernel of  $f_0$*  is a sequence

$$(f_1, f_2, \dots, f_n) : A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_{n+1}$$

such that for all  $k = 1, 2, \dots, n-1$ , the morphism  $f_k$  is a weak cokernel of  $f_{k-1}$ , and  $f_n$  is a cokernel of  $f_{n-1}$ . In this case, we say the sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \dots \xrightarrow{f_n} A_{n+1} \quad (2.1)$$

is *right  $n$ -exact*.

**Remark 2.5.** When we say  $n$ -cokernel we always means that  $n$  is a positive integer. We note that the notion of 1-cokernel is the same as cokernel. we can define  *$n$ -kernel* and *left  $n$ -exact sequence* dually. The sequence (2.1) is called  *$n$ -exact* if it is both right  $n$ -exact and left  $n$ -exact.

**Definition 2.6.** Let  $\mathcal{X}$  be a subcategory of  $\mathcal{C}$  and  $f_0 : A_0 \rightarrow A_1$  a morphism in  $\mathcal{C}$ . We say  $f_0$  has a *special  $n$ -cokernel with respect to  $\mathcal{X}$* , if  $f_0$  has an  $n$ -cokernel

$$(f_1, f_2, \dots, f_{n-1}, f_n) : A_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} A_{n+1}$$

with  $X_i \in \mathcal{X}$ .

**Definition 2.7.** ([8, Definition 2.11]) Let  $\mathcal{C}$  be an additive category,  $A = A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} A_n$  a complex and  $g : A_0 \rightarrow B_0$  a morphism in  $\mathcal{C}$ . An  *$n$ -pushout*

diagram of  $A$  along  $g$  is a morphism of complexes

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n \\ \downarrow g & & \downarrow h_1 & & & & \downarrow h_{n-1} & & \downarrow h_n \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & \cdots & \xrightarrow{g_{n-2}} & B_{n-1} & \xrightarrow{g_{n-1}} & B_n \end{array}$$

such that the mapping cone

$$A_0 \xrightarrow{\begin{pmatrix} -f_0 \\ g \end{pmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ h_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ h_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ h_{n-1} & g_{n-2} \end{pmatrix}} A_n \oplus B_{n-1} \xrightarrow{\begin{pmatrix} h_n & g_{n-1} \end{pmatrix}} B_n$$

is right  $n$ -exact.

The following lemma is useful in the proof of Theorem 3.4.

**Lemma 2.8.** *Let*

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \parallel & & \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n & & \downarrow h_{n+1} \\ A_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} \end{array}$$

be a commutative diagram of right  $n$ -exact sequences. Then

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n & & \downarrow h_{n+1} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} \end{array}$$

is an  $n$ -pushout diagram, in other words,

$$A_1 \xrightarrow{d_0} A_2 \oplus B_1 \xrightarrow{d_1} A_3 \oplus B_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} A_{n+1} \oplus B_n \xrightarrow{d_n} B_{n+1}$$

is a right  $n$ -exact sequence, where  $d_0 = \begin{pmatrix} -f_1 \\ h_1 \end{pmatrix}$ ,  $d_i = \begin{pmatrix} -f_{i+1} & 0 \\ h_{i+1} & g_i \end{pmatrix}$  ( $i = 1, 2, \dots, n-1$ ) and  $d_n = (h_{n+1} \ g_n)$ .

*Proof.* It is easy to check that  $d_{i+1}d_i = 0, i = 0, 1, \dots, n-1$ . Let  $s_0 = (a_1 \ b_1) : A_2 \oplus B_1 \rightarrow C_1$  be a morphism such that  $s_0d_0 = 0$ , then  $a_1f_1 = b_1h_1$ . Note that  $b_1g_0 = b_1h_1f_0 = a_1f_1f_0 = 0$ , there exists a morphism  $v_1 : B_2 \rightarrow C_1$  such that  $b_1 = v_1g_1$  since  $g_1$  is a weak cokernel of  $g_0$ . Now we have  $(a_1 - v_1h_2)f_1 = b_1h_1 - v_1g_1h_1 = 0$ , thus there exists a morphism  $u_1 : A_3 \rightarrow C_1$  such that  $a_1 - v_1h_2 = u_1f_2$  since  $f_2$  is a weak cokernel of  $f_1$ . Hence  $s_0 = (a_1 \ b_1) = (-u_1 \ v_1) \begin{pmatrix} -f_2 & 0 \\ h_2 & g_1 \end{pmatrix}$ . This shows that  $d_1$  is a weak cokernel of  $d_0$ .

Let  $s_{k-1} = (a_k \ b_k) : A_{k+1} \oplus B_k \rightarrow C_k$  be a morphism such that  $s_{k-1}d_{k-1} = 0$  where  $k = 2, 3, \dots, n-1$ . Then  $a_kf_k = b_kh_k$  and  $b_kg_{k-1} = 0$ . There exists a morphism  $v_k : B_{k+1} \rightarrow C_k$  such that  $b_k = v_kg_k$  since  $g_k$  is a weak cokernel of  $g_{k-1}$ . Note that  $(a_k - v_kh_{k+1})f_k = b_kh_k - v_kg_kh_k = 0$ , there exists a morphism  $u_k : A_{k+2} \rightarrow C_k$  such that  $a_k - v_kh_{k+1} = u_kf_{k+1}$  since  $f_{k+1}$  is a weak cokernel of  $f_k$ . Hence  $s_{k-1} = (a_k \ b_k) = (-u_k \ v_k) \begin{pmatrix} -f_{k+1} & 0 \\ h_{k+1} & g_k \end{pmatrix}$ . This shows that  $d_k$  is a weak cokernel of  $d_{k-1}$ .

It remains to show that  $d_{n-1}$  is a cokernel of  $d_n$ . Let  $s_{n-1} = (a_n \ b_n) : A_{n+1} \oplus B_n \rightarrow C_n$  be a morphism such that  $s_{n-1}d_{n-1} = 0$ . Then  $a_nf_n = b_nh_n$  and  $b_ng_{n-1} = 0$ . There exists a morphism  $v_n : B_{n+1} \rightarrow C_n$  such that  $b_n = v_ng_n$ . Now we have  $(b_nh_n)f_{n-1} = b_ng_{n-1}h_{n-1} = v_ng_ng_{n-1}h_{n-1} = 0$ ,  $a_nf_n = b_nh_n$  and  $(v_nh_{n+1})f_n = v_ng_nh_n = b_nh_n$ , which implies that  $a_n = v_nh_{n+1}$  since  $f_n$  is a cokernel of  $f_{n-1}$ . So  $s_{n-1} = (a_n \ b_n) = v_n(h_{n+1} \ g_n)$ , and  $d_n$  is a weak cokernel of  $d_{n-1}$ . Note that  $g_n$  is an epimorphism thus so is  $d_n = (h_{n+1}, g_n)$ . The proof is completed.  $\square$

### 3. MAIN RESULTS

Assume that  $\mathcal{C}$  is an additive category and  $\mathcal{X}$  a covariantly finite subcategory.

**Proposition 3.1.** *If any left  $\mathcal{X}$ -approximation has a special  $n$ -cokernel with respect to  $\mathcal{X}$ , then there is an additive functor  $\Sigma : \mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$ .*

*Proof.* For any object  $A \in \mathcal{C}$ , fix a right  $n$ -exact sequence

$$A \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} B$$

where  $\alpha_0$  is a left  $\mathcal{X}$ -approximation of  $A$  and  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  is a special  $n$ -cokernel of  $\alpha_0$ . For any morphism  $f : A \rightarrow A'$ , since  $\alpha_0$  is a left  $\mathcal{X}$ -approximation, we have the following commutative diagram

$$\begin{array}{ccccccccccc} A & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & B \\ \downarrow f & & \downarrow x_1 & & \downarrow x_2 & & & & \downarrow x_n & & \downarrow g \\ A' & \xrightarrow{\alpha'_0} & X'_1 & \xrightarrow{\alpha'_1} & X'_2 & \xrightarrow{\alpha'_2} & \cdots & \xrightarrow{\alpha'_{n-1}} & X'_n & \xrightarrow{\alpha'_n} & B' \end{array}$$

Define a functor  $\Sigma : \mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$  such that  $\Sigma A = B$  and  $\Sigma \underline{f} = \underline{g}$ . It is easy to see that  $\Sigma$  is a well defined additive functor by Lemma 2.3.  $\square$

**Definition 3.2.** Let

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

be a right  $n$ -exact sequence where  $f_0$  is  $\mathcal{X}$ -monic. Then there exists the following commutative diagram

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \parallel & & \downarrow a_1 & & \downarrow a_2 & & & & \downarrow a_n & & \downarrow a_{n+1} \\ A_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma A_0. \end{array} \quad (3.1)$$

The  $n$ - $\Sigma$ -sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0$$

is called a *standard right  $(n+2)$ -angle* in  $\mathcal{C}/\mathcal{X}$ . We define  $\Theta$  the class of  $n$ - $\Sigma$ -sequences which are isomorphic to standard right  $(n+2)$ -angles.

**Remark 3.3.** We add the sign  $(-1)^n$  in the last morphism of standard right  $(n+2)$ -angles. In the dual case, maybe we don't need the sign in the first morphism of standard left  $(n+2)$ -angles. We will see the difference in Corollary 3.7(c). We also can see [7, Lemma 2.7] and [8, Lemma 5.10] for the sign.

**Lemma 3.4.** *Let*

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n & & \downarrow h_{n+1} \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} \end{array}$$

be a commutative diagram of right  $n$ -exact sequences, where  $f_0$  and  $g_0$  are  $\mathcal{X}$ -monic. Then we have the following commutative diagram of standard right  $(n+2)$ -angles.

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} & \xrightarrow{(-1)^n a_{n+1}} & \Sigma A_0 \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n & & \downarrow h_{n+1} & & \downarrow \Sigma h_0 \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} & \xrightarrow{(-1)^n b_{n+1}} & \Sigma B_0 \end{array}$$

*Proof.* We only need to show that  $\Sigma h_0 \cdot a_{n+1} = b_{n+1} \cdot h_{n+1}$ . By the constructions of standard  $(n+2)$ -angles and the morphism  $\Sigma h_0$ , we have the following two commutative diagrams

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \parallel & & \downarrow a_1 & & \downarrow a_2 & & & & \downarrow a_n & & \downarrow a_{n+1} \\ A_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma A_0 \\ \downarrow h_0 & & \downarrow x_1 & & \downarrow x_2 & & & & \downarrow x_n & & \downarrow i_0 \\ B_0 & \xrightarrow{\beta_0} & X'_1 & \xrightarrow{\beta_1} & X'_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & X'_n & \xrightarrow{\beta_n} & \Sigma B_0, \end{array}$$

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 & & & & \downarrow h_n & & \downarrow h_{n+1} \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} \\ \parallel & & \downarrow b_1 & & \downarrow b_2 & & & & \downarrow b_n & & \downarrow b_{n+1} \\ B_0 & \xrightarrow{\beta_0} & X'_1 & \xrightarrow{\beta_1} & X'_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & X'_n & \xrightarrow{\beta_n} & \Sigma B_0 \end{array}$$

where  $\Sigma h_0 = i_0$ . Lemma 2.3 implies that  $\Sigma h_0 \cdot a_{n+1} = b_{n+1} \cdot h_{n+1}$ .  $\square$

Now we can state and prove our first main result.

**Theorem 3.5.** *Let  $\mathcal{C}$  be an additive category and  $\mathcal{X}$  a covariantly finite subcategory. If any  $\mathcal{X}$ -monic has an  $n$ -cokernel and any left  $\mathcal{X}$ -approximation has a special  $n$ -cokernel with respect to  $\mathcal{X}$ , then the quotient category  $\mathcal{C}/\mathcal{X}$  is a right  $(n+2)$ -angulated category with respect to the functor  $\Sigma$  defined in Proposition 3.1 and  $(n+2)$ -angles defined in Definition 3.2.*

*Proof.* It is easy to see that the class of  $(n+2)$ -angles  $\Theta$  is closed under direct sums and direct summands, so (RN1)(a) is satisfied. For any object  $A \in \mathcal{C}$ , the identity

morphism of  $A$  is  $\mathcal{X}$ -monic. The commutative diagram

$$\begin{array}{ccccccccccc} A & \xrightarrow{1_A} & A & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\ \parallel & & \downarrow \alpha_0 & & \downarrow 0 & & & & \downarrow 0 & & \downarrow 0 \\ A & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma A \end{array}$$

shows that

$$A \xrightarrow{1_A} A \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma A$$

belongs to  $\Phi$ . Thus (RN1)(b) is satisfied.

For any morphism  $f : A \rightarrow B$  in  $\mathcal{C}$ , the morphism  $\begin{pmatrix} f \\ \alpha_0 \end{pmatrix} : A \rightarrow B \oplus X_1$  is  $\mathcal{X}$ -monic, where  $\alpha_0$  is a left  $\mathcal{X}$ -approximation. Let  $(f_1, f_2, \dots, f_n)$  be an  $n$ -cokernel of  $\begin{pmatrix} f \\ \alpha_0 \end{pmatrix}$ , then we have the following commutative diagram

$$\begin{array}{ccccccccccc} A & \xrightarrow{\begin{pmatrix} f \\ \alpha_0 \end{pmatrix}} & B \oplus X_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \parallel & & \downarrow a_1 & & \downarrow a_2 & & & & \downarrow a_n & & \downarrow a_{n+1} \\ A & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma A. \end{array}$$

Thus

$$A \xrightarrow{f} B \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A$$

is a right  $(n+2)$ -angle. So (RN1)(c) is satisfied.

(RN2) It is easy to see that it suffices to consider the case of standard  $(n+2)$ -angles. Let

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0$$

be a standard  $(n+2)$ -angle induced by the commutative diagram (3.1). By Lemma 2.8, we have a right  $n$ -exact sequence

$$A_1 \xrightarrow{\begin{pmatrix} -f_1 \\ a_1 \end{pmatrix}} A_2 \oplus X_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ a_2 & \alpha_1 \end{pmatrix}} A_3 \oplus X_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ a_3 & \alpha_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ a_n & \alpha_{n-1} \end{pmatrix}} A_{n+1} \oplus X_n \xrightarrow{\begin{pmatrix} a_{n+1} & \alpha_n \end{pmatrix}} \Sigma A_0.$$

We claim that the morphism  $\begin{pmatrix} -f_1 \\ a_1 \end{pmatrix} : A_1 \rightarrow A_2 \oplus X_1$  is  $\mathcal{X}$ -monic. For any morphism  $g : A_1 \rightarrow X$  with  $X \in \mathcal{X}$ , there exists a morphism  $h : X_1 \rightarrow X$  such that  $gf_0 = h\alpha_0$  since  $\alpha_0$  is  $\mathcal{X}$ -monic. Note that  $(g - ha_1)f_0 = gf_0 - h\alpha_0 = 0$ , which implies that there exists a morphism  $k : A_2 \rightarrow X$  such that  $g - ha_1 = kf_1$ . Thus we have  $g = \begin{pmatrix} -k & h \end{pmatrix} \begin{pmatrix} -f_1 \\ a_1 \end{pmatrix}$ . The following commutative diagram

$$\begin{array}{ccccccccccc} A_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma A_0 \\ \downarrow f_0 & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \parallel \\ A_1 & \xrightarrow{\begin{pmatrix} -f_1 \\ a_1 \end{pmatrix}} & A_2 \oplus X_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ a_2 & \alpha_1 \end{pmatrix}} & A_3 \oplus X_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ a_3 & \alpha_2 \end{pmatrix}} & \cdots & \xrightarrow{\begin{pmatrix} -f_n & 0 \\ a_n & \alpha_{n-1} \end{pmatrix}} & A_{n+1} \oplus X_n & \xrightarrow{\begin{pmatrix} a_{n+1} & \alpha_n \end{pmatrix}} & \Sigma A_0 \\ \parallel & & \downarrow b_1 & & \downarrow b_2 & & & & \downarrow b_n & & \downarrow b_{n+1} \\ A_1 & \xrightarrow{\beta_0} & X'_1 & \xrightarrow{\beta_1} & X'_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & X'_n & \xrightarrow{\beta_n} & \Sigma A_1 \end{array}$$

implies that  $\underline{b_{n+1}} = \Sigma f_0$  and

$$A_1 \xrightarrow{-f_1} A_2 \xrightarrow{-f_2} A_3 \xrightarrow{-f_3} \cdots \xrightarrow{-f_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma A_0 \xrightarrow{(-1)^n \Sigma f_0} \Sigma A_1 \quad (3.2)$$



is a right  $(n+2)$ -angle. Therefore

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0 \xrightarrow{(-1)^n \Sigma f_0} \Sigma A_1 \quad (3.3)$$

belongs to  $\Theta$  since the  $(n+2)$ - $\Sigma$ -sequences (3.2) and (3.3) are isomorphic.

(RN3) We only consider standard right  $(n+2)$ -angles. Suppose that there is a commutative diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0 \\ \downarrow h_0 & & \downarrow h_1 & & & & \downarrow \Sigma h_0 \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_n} B_{n+1} \xrightarrow{(-1)^n b_{n+1}} \Sigma B_0 \end{array} \quad (3.4)$$

with rows standard right  $(n+2)$ -angles. Since  $h_1 \cdot f_0 = g_0 \cdot h_0$  holds,  $h_1 f_0 - g_0 h_0$  factors through some object  $X$  in  $\mathcal{X}$ . Assume that  $h_1 f_0 - g_0 h_0 = ba$ , where  $a : A_0 \rightarrow X$  and  $b : X \rightarrow B_1$ . Since  $f_0$  is  $\mathcal{X}$ -monic, there exists a morphism  $c : A_1 \rightarrow X$  such that  $a = cf_0$ . Note that  $(h_1 - bc)f_0 = g_0 h_0$ , we have the following commutative diagram of right  $n$ -exact sequences

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_n} A_{n+1} \\ \downarrow h_0 & & \downarrow h_1 - bc & & \downarrow h_2 & & \downarrow h_{n+1} \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_n} B_{n+1} \end{array} \quad (3.5)$$

by the factorization property of weak cokernels. The diagram (3.4) can be completed to a morphism of right  $(n+2)$ -angles follows from diagram (3.5) together with Lemma 3.4.

(RN4) It is enough to consider the case of standard right  $(n+2)$ -angles. Let

$$\begin{array}{l} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0 \\ A_0 \xrightarrow{\varphi_1 f_0} B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} B_{n+1} \xrightarrow{(-1)^n b_{n+1}} \Sigma A_0 \\ A_1 \xrightarrow{\varphi_1} B_1 \xrightarrow{h_1} C_2 \xrightarrow{h_2} \cdots \xrightarrow{h_n} C_{n+1} \xrightarrow{(-1)^n c_{n+1}} \Sigma A_1 \end{array}$$

be three standard right  $(n+2)$ -angles, where  $f_0$  and  $\varphi_1$  are  $\mathcal{X}$ -monic, so that  $\varphi_1 f_0$  is  $\mathcal{X}$ -monic too. By the factorization property of weak cokernels there exist morphisms  $\varphi_i : A_i \rightarrow B_i$  ( $i = 2, \dots, n+1$ ), such that we have the following commutative diagram of right  $n$ -exact sequences

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \\ \parallel & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_n \\ A_0 & \xrightarrow{\varphi_1 f_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_{n-1}} B_n \xrightarrow{g_n} B_{n+1} \end{array} \quad (3.6)$$

By Lemma 3.4, we obtain the following commutative diagram of right  $(n+2)$ -angles

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0 \\ \parallel & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_n \\ A_0 & \xrightarrow{\varphi_1 f_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \xrightarrow{g_{n-1}} B_n \xrightarrow{g_n} B_{n+1} \xrightarrow{(-1)^n b_{n+1}} \Sigma A_0. \end{array}$$

Diagram (3.6) and Lemma 2.8 implies that

$$A_1 \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} A_{n+1} \oplus B_n \xrightarrow{(\varphi_{n+1} \ g_n)} B_{n+1}$$

is a right  $n$ -exact sequence. There exist morphisms  $\phi_i : A_i \rightarrow C_{i-1}$  ( $i = 3, 4, \dots, n+1$ ) and  $\psi_j : B_j \rightarrow C_j$  ( $j = 2, 3, \dots, n+1$ ) such that the following diagram is commutative

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} & A_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} & \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} A_{n+1} \oplus B_n \xrightarrow{(\varphi_{n+1} \ g_n)} B_{n+1} \\ \parallel & & \parallel & & \downarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{pmatrix} & \parallel \\ A_1 & \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} & A_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} f_3 & 0 \\ -\varphi_3 & g_2 \end{pmatrix}} & \cdots \xrightarrow{\begin{pmatrix} f_n & 0 \\ (-1)^n \varphi_n & g_{n-1} \end{pmatrix}} A_{n+1} \oplus B_n \xrightarrow{\begin{pmatrix} (-1)^{n+1} \varphi_{n+1} & g_n \end{pmatrix}} B_{n+1} \\ \parallel & & \downarrow (0 \ 1) & & \downarrow (\phi_3 \ \psi_2) & & \downarrow (\phi_{n+1} \ \psi_n) & \parallel \\ A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & \cdots \xrightarrow{h_{n-1}} & C_n \xrightarrow{h_n} C_{n+1} \end{array}$$

where the second row is a right  $n$ -exact sequence since it is isomorphic to the first row. By Lemma 2.8 again, we get a right  $n$ -exact sequence

$$\begin{aligned} A_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ -\varphi_2 & -g_1 \\ 0 & 1 \end{pmatrix}} A_3 \oplus B_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_3 & 0 & 0 \\ \varphi_3 & -g_2 & 0 \\ \phi_3 & \psi_2 & h_1 \end{pmatrix}} A_4 \oplus B_3 \oplus C_2 \\ & \xrightarrow{\begin{pmatrix} -f_4 & 0 & 0 \\ -\varphi_4 & -g_3 & 0 \\ \phi_4 & \psi_3 & h_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 & 0 \\ (-1)^{n+1} \varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{pmatrix}} A_{n+1} \oplus B_n \oplus C_{n-1} \\ & \xrightarrow{\begin{pmatrix} (-1)^{n+2} \varphi_{n+1} & -g_n & 0 \\ \phi_{n+1} & \psi_n & h_{n-1} \end{pmatrix}} B_{n+1} \oplus C_n \xrightarrow{(\psi_{n+1} \ h_n)} C_{n+1}. \end{aligned} \quad (3.7)$$

The following commutative diagram

$$\begin{array}{ccccc} A_2 & \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \\ \phi_3 & \psi_2 \end{pmatrix}} & A_4 \oplus B_3 \oplus C_2 \\ \downarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \parallel \\ A_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ -\varphi_2 & -g_1 \\ 0 & 1 \end{pmatrix}} & A_3 \oplus B_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} -f_3 & 0 & 0 \\ \varphi_3 & -g_2 & 0 \\ \phi_3 & \psi_2 & h_1 \end{pmatrix}} & A_4 \oplus B_3 \oplus C_2 \\ \downarrow \begin{pmatrix} -1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_1 \end{pmatrix} & & \parallel \\ A_2 & \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \\ \phi_3 & \psi_2 \end{pmatrix}} & A_4 \oplus B_3 \oplus C_2 \end{array}$$

shows that

$$\begin{aligned} A_2 & \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \\ \phi_3 & \psi_2 \end{pmatrix}} A_4 \oplus B_3 \oplus C_2 \xrightarrow{\begin{pmatrix} -f_4 & 0 & 0 \\ -\varphi_4 & -g_3 & 0 \\ \phi_4 & \psi_3 & h_3 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 & 0 \\ (-1)^{n+1} \varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{pmatrix}} \\ & A_{n+1} \oplus B_n \oplus C_{n-1} \xrightarrow{\begin{pmatrix} (-1)^{n+2} \varphi_{n+1} & -g_n & 0 \\ \phi_{n+1} & \psi_n & h_{n-1} \end{pmatrix}} B_{n+1} \oplus C_n \xrightarrow{(\psi_{n+1} \ h_n)} C_{n+1} \end{aligned} \quad (3.8)$$

is a direct summand of (3.7), thus it is a right  $n$ -exact sequence. We claim that the morphism  $\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix} : A_2 \rightarrow A_3 \oplus B_2$  is  $\mathcal{X}$ -monic. In fact, for any morphism  $a : A_2 \rightarrow X$  with  $X \in \mathcal{X}$ , there exists a morphism  $b : B_1 \rightarrow X$  such that  $af_1 = b\varphi_1$  since  $\varphi_1$  is  $\mathcal{X}$ -monic. Note that  $b(\varphi_1 f_0) = af_1 f_0 = 0$ , there exists a morphism  $c : B_2 \rightarrow X$

such that  $b = cg_1$ . Since  $(a - c\varphi_2)f_1 = b\varphi_1 - cg_1\varphi_1 = 0$ , there exists a morphism  $d : A_3 \rightarrow X$  such that  $a - c\varphi_2 = df_2$ . Thus  $a = (d \ c) \begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}$ . Therefore (3.8) induces a standard right  $(n+2)$ -angle

$$\begin{aligned} A_2 \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} A_3 \oplus B_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \end{pmatrix}} A_4 \oplus B_3 \oplus C_2 \xrightarrow{\begin{pmatrix} -f_4 & 0 & 0 \\ -\varphi_4 & -g_3 & 0 \\ \phi_4 & \psi_3 & h_3 \end{pmatrix}} \dots \xrightarrow{\begin{pmatrix} -f_n & 0 & 0 \\ (-1)^{n+1}\varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{pmatrix}} \\ A_{n+1} \oplus B_n \oplus C_{n-1} \xrightarrow{\begin{pmatrix} (-1)^{n+2}\varphi_{n+1} & -g_n & 0 \\ \phi_{n+1} & \psi_n & h_{n-1} \end{pmatrix}} B_{n+1} \oplus C_n \xrightarrow{(\psi_{n+1} \ h_n)} C_{n+1} \xrightarrow{(-1)^n d_{n+1}} \Sigma A_2. \end{aligned}$$

By Lemma 3.4 the following commutative diagram of right  $(n+2)$ -angles

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\varphi_1 f_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \dots \\ \downarrow f_0 & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \\ A_1 & \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} & A_2 \oplus B_1 & \xrightarrow{\begin{pmatrix} f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} f_3 & 0 \\ -\varphi_3 & g_2 \end{pmatrix}} & \dots \\ \parallel & & \downarrow \begin{pmatrix} 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} \phi_3 & \psi_2 \end{pmatrix} & & \\ A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{h_1} & C_2 & \xrightarrow{h_2} & \dots \\ \downarrow f_1 & & \downarrow \begin{pmatrix} 0 \\ g_1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \\ A_2 & \xrightarrow{\begin{pmatrix} f_2 \\ \varphi_2 \end{pmatrix}} & A_3 \oplus B_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & -g_2 \end{pmatrix}} & A_4 \oplus B_3 \oplus C_2 & \xrightarrow{\begin{pmatrix} -f_4 & 0 & 0 \\ \varphi_4 & -g_3 & 0 \\ \phi_4 & \psi_3 & h_3 \end{pmatrix}} & \dots \end{array}$$
  

$$\begin{array}{ccccccc} \xrightarrow{g_{n-1}} & B_n & \xrightarrow{g_n} & B_{n+1} & \xrightarrow{(-1)^n b_{n+1}} & \Sigma A_0 & \\ & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \parallel & & \downarrow \Sigma f_0 & \\ \xrightarrow{\begin{pmatrix} f_n & 0 \\ (-1)^n \varphi_n & g_{n-1} \end{pmatrix}} & A_{n+1} \oplus B_n & \xrightarrow{((-1)^{n+1} \varphi_{n+1} \ g_n)} & B_{n+1} & \xrightarrow{(-1)^n e_{n+1}} & \Sigma A_1 & \\ & \downarrow \begin{pmatrix} \phi_{n+1} & \psi_2 \end{pmatrix} & & \downarrow \psi_{n+1} & & \parallel & \\ \xrightarrow{h_{n-1}} & C_n & \xrightarrow{h_n} & C_{n+1} & \xrightarrow{(-1)^n c_{n+1}} & \Sigma A_1 & \\ & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \parallel & & \downarrow \Sigma f_1 & \\ \xrightarrow{\begin{pmatrix} (-1)^{n+2} \varphi_{n+1} & -g_n & 0 \\ \phi_{n+1} & \psi_n & h_{n-1} \end{pmatrix}} & B_{n+1} \oplus C_n & \xrightarrow{(\psi_{n+1} \ h_n)} & C_{n+1} & \xrightarrow{(-1)^n d_{n+1}} & \Sigma A_2 & \end{array}$$

shows that  $c_{n+1} \cdot \psi_{n+1} = \Sigma f_0 \cdot b_{n+1}$  and  $d_{n+1} = \Sigma f_1 \cdot c_{n+1}$ . This is what we wanted. We finish the proof.  $\square$

**Remarks 3.6.** (a) If  $n = 1$ , then the condition “any left  $\mathcal{X}$ -approximation has a special  $n$ -cokernel with respect to  $\mathcal{X}$ ” is trivial. Thus Theorem 3.5 recover Beligiannis-Marmaridis’s result [2, Theorem 2.12].

(b) From the proof of Theorem 3.5 we see that the condition “any left  $\mathcal{X}$ -approximation has a special  $n$ -cokernel with respect to  $\mathcal{X}$ ” can be weakened as “for any object  $A \in \mathcal{C}$  there exists a left  $\mathcal{X}$ -approximation  $f : A \rightarrow X$  such that  $f$  has a special  $n$ -cokernel with respect to  $\mathcal{X}$ ”. The condition “any  $\mathcal{X}$ -monic has an  $n$ -cokernel” is used to prove (RN1)(c), which is redundant in some special case. See Corollary 3.7(a) for details.

We recall some definitions given by Jasso [8]. Let  $\mathcal{A}$  be an  $n$ -abelian category and  $(\mathcal{C}, \mathcal{S})$  an  $n$ -exact subcategory, where  $\mathcal{S}$  is a class of admissible  $n$ -exact sequences satisfying some axioms similar to exact category. An object  $I \in \mathcal{C}$  is called  $\mathcal{S}$ -*injective* if for any admissible monomorphism  $f : A \rightarrow B$ , the sequence  $\mathcal{C}(B, I) \xrightarrow{\mathcal{C}(f, I)} \mathcal{C}(A, I) \rightarrow 0$  is exact. Denote by  $\mathcal{I}$  the subcategory of  $\mathcal{S}$ -injectives. We say that  $(\mathcal{C}, \mathcal{S})$  has enough  $\mathcal{S}$ -injectives if for any object  $A \in \mathcal{C}$ , there exists an admissible  $n$ -exact sequence  $A \rightarrowtail I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_n \twoheadrightarrow B$  with  $I_i \in \mathcal{I}$ . We can define the notion of  $\mathcal{S}$ -*projective* and *having enough  $\mathcal{S}$ -projectives* dually. Denote by  $\mathcal{P}$  the subcategory of  $\mathcal{S}$ -projectives. We say that  $(\mathcal{C}, \mathcal{S})$  is *Frobenius* if it has enough  $\mathcal{S}$ -injectives, has enough  $\mathcal{S}$ -projectives and if  $\mathcal{S}$ -injectives and  $\mathcal{S}$ -projectives coincide. Now we can derive a theorem of Jasso [8, Theorem 5.11].

**Corollary 3.7.** *Let  $\mathcal{A}$  be an  $n$ -abelian category and  $(\mathcal{C}, \mathcal{S})$  an  $n$ -exact subcategory.*

(a) *If  $(\mathcal{C}, \mathcal{S})$  has enough  $\mathcal{S}$ -injectives, then the quotient  $\mathcal{C}/\mathcal{I}$  is a right  $(n+2)$ -angulated category.*

(b) *If  $(\mathcal{C}, \mathcal{S})$  has enough  $\mathcal{S}$ -projectives, then the quotient  $\mathcal{C}/\mathcal{P}$  is a left  $(n+2)$ -angulated category.*

(c) *If  $(\mathcal{C}, \mathcal{S})$  is Frobenius, then the quotient  $\mathcal{C}/\mathcal{I}$  is an  $(n+2)$ -angulated category.*

*Proof.* (a) It is easy to see that  $\mathcal{I}$  is a covariantly finite subcategory of  $\mathcal{C}$ . By the definition of having enough  $\mathcal{S}$ -injectives and Remarks 3.6 (b) we only need to prove (RN1)(c). Let  $f : A \rightarrow B$  be a morphism in  $\mathcal{C}$ , we have the following  $n$ -pushout diagram by the axiom of  $n$ -exact category

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_0} & I_1 & \xrightarrow{\alpha_1} & I_2 & \xrightarrow{\alpha_2} & \cdots \xrightarrow{\alpha_{n-1}} I_n \xrightarrow{\alpha_n} \Sigma A \\ \downarrow f & & \downarrow a_1 & & \downarrow a_2 & & \downarrow a_n \\ B & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \xrightarrow{f_{n-1}} A_n \end{array}$$

Then

$$A \xrightarrow{\begin{pmatrix} -\alpha_0 \\ f \end{pmatrix}} I_1 \oplus B \xrightarrow{\begin{pmatrix} -\alpha_1 & 0 \\ a_1 & f_0 \end{pmatrix}} I_2 \oplus A_1 \xrightarrow{\begin{pmatrix} -\alpha_2 & 0 \\ a_2 & f_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -\alpha_{n-1} & 0 \\ a_{n-1} & f_{n-2} \end{pmatrix}} I_n \oplus A_{n-1} \xrightarrow{\begin{pmatrix} a_n & f_{n-1} \end{pmatrix}} A_n \quad (3.9)$$

is a right  $n$ -exact sequence. Note that the morphism  $\begin{pmatrix} -\alpha_0 \\ f \end{pmatrix}$  is  $\mathcal{I}$ -monic, thus (3.9) induces a right  $(n+2)$ -angle whose first morphism is  $\underline{f}$ .

(b) is dual to (a). (c) follows from (a) together with Remarks 2.2(c). In fact, since  $(\mathcal{C}, \mathcal{S})$  is Frobenius it is easy to see that the endofunctor  $\Sigma : \mathcal{C}/\mathcal{I} \rightarrow \mathcal{C}/\mathcal{I}$  is an equivalence. We note that a right  $n$ -exact sequence in  $\mathcal{C}$  is admissible  $n$ -exact if and only if the first morphism is  $\mathcal{I}$ -monic. Since  $(\mathcal{C}, \mathcal{S})$  is Frobenius, we have the following commutative diagram of admissible  $n$ -exact sequences

$$\begin{array}{ccccccccccc} \Sigma^{-1} A_{n+1} & \xrightarrow{\beta_0} & P_1 & \xrightarrow{\beta_1} & P_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & P_n & \xrightarrow{\beta_n} & A_{n+1} \\ \downarrow b_{n+1} & & \downarrow b_n & & \downarrow b_{n-1} & & & & \downarrow b_1 & & \parallel \\ A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & A_{n+1} \\ \parallel & & \downarrow a_1 & & \downarrow a_2 & & & & \downarrow a_n & & \downarrow a_{n+1} \\ A_0 & \xrightarrow{\alpha_0} & I_1 & \xrightarrow{\alpha_1} & I_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & I_n & \xrightarrow{\alpha_n} & \Sigma A_0, \end{array}$$

which implies that

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \xrightarrow{(-1)^n a_{n+1}} \Sigma A_0$$

is a standard right  $(n+2)$ -angle in  $\mathcal{C}/\mathcal{I}$  if and only if

$$\Sigma^{-1} A_{n+1} \xrightarrow{\Sigma^{-1} a_{n+1}} A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1}$$

is a standard left  $(n+2)$ -angle in  $\mathcal{C}/\mathcal{I}$ . Thus the class of right  $(n+2)$ -angles and the class of left  $(n+2)$ -angles coincide.  $\square$

From now on, let  $n \geq 3$ . In the rest of this section we consider the right  $n$ -angulated categories arising from covariantly finite subcategories of  $n$ -angulated categories.

**Theorem 3.8.** *Let  $\mathcal{C}$  be an  $n$ -angulated category and  $\mathcal{X}$  a covariantly finite subcategory. If for any object  $A \in \mathcal{C}$  there exists an  $n$ -angle*

$$A \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} X_{n-2} \xrightarrow{\alpha_{n-1}} B \xrightarrow{\alpha_n} \Sigma A$$

where  $X_i \in \mathcal{X}$  and  $\alpha_1$  is a left  $\mathcal{X}$ -approximation. Then the quotient  $\mathcal{C}/\mathcal{X}$  is a right  $n$ -angulated category.

*Proof.* For any object  $A \in \mathcal{C}$ , fix an  $n$ -angle

$$A \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} X_{n-2} \xrightarrow{\alpha_{n-1}} B \xrightarrow{\alpha_n} \Sigma A$$

where  $X_i \in \mathcal{X}$  and  $\alpha_1$  is a left  $\mathcal{X}$ -approximation. For any morphism  $f : A \rightarrow A'$ , since  $\alpha_1$  is a left  $\mathcal{X}$ -approximation, we have the following commutative diagram

$$\begin{array}{ccccccccccc} A & \xrightarrow{\alpha_1} & X_1 & \xrightarrow{\alpha_2} & X_2 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-2} & \xrightarrow{\alpha_{n-1}} & B & \xrightarrow{\alpha_n} & \Sigma A \\ \downarrow f & & \downarrow x_1 & & \downarrow x_2 & & & & \downarrow x_{n-2} & & \downarrow g & & \downarrow \Sigma f \\ A' & \xrightarrow{\alpha'_1} & X'_1 & \xrightarrow{\alpha'_2} & X'_2 & \xrightarrow{\alpha'_3} & \cdots & \xrightarrow{\alpha'_{n-2}} & X'_{n-2} & \xrightarrow{\alpha'_{n-1}} & B' & \xrightarrow{\alpha'_n} & \Sigma A' \end{array}$$

Define a functor  $T : \mathcal{C}/\mathcal{X} \rightarrow \mathcal{C}/\mathcal{X}$  such that  $TA = B$  and  $T\underline{f} = \underline{g}$ . It is easy to see that  $T$  is a well defined additive functor. Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} \Sigma A_1$$

be an  $n$ -angle where  $f_1$  is  $\mathcal{X}$ -monic. Then there exists the following commutative diagram

$$\begin{array}{ccccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & A_{n-1} & \xrightarrow{f_{n-1}} & A_n & \xrightarrow{f_n} & \Sigma A_1 \\ \parallel & & \downarrow a_2 & & \downarrow a_3 & & & & \downarrow a_{n-1} & & \downarrow a_n & & \parallel \\ A_1 & \xrightarrow{\alpha_1} & X_1 & \xrightarrow{\alpha_2} & X_2 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-2} & \xrightarrow{\alpha_{n-1}} & TA_1 & \xrightarrow{\alpha_n} & \Sigma A_1 \end{array}$$

The  $n$ - $T$ -sequence

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{a_n} TA_1$$

is called a standard right  $n$ -angle in  $\mathcal{C}/\mathcal{X}$ . Denote by  $\Theta$  the class of  $n$ - $T$ -sequences which are isomorphic to standard right  $n$ -angles. We can show that  $(\mathcal{C}/\mathcal{X}, T, \Theta)$  is a right  $n$ -angulated category. Since the proof is similar to [9, Theorem 3.8], we omit it.  $\square$

In particular, if  $n = 3$ , then we get the following well known result.

**Corollary 3.9.** *Let  $\mathcal{C}$  be a triangulated category and  $\mathcal{X}$  a covariantly finite subcategory. Then the quotient  $\mathcal{C}/\mathcal{X}$  is a right triangulated category.*

The following corollary follows immediately from Theorem 3.8 and its dual.

**Corollary 3.10.** *(cf. [9, Theorem 3.8]) Let  $\mathcal{C}$  be an  $n$ -angulated category and  $\mathcal{X}$  a functorially finite subcategory of  $\mathcal{C}$ . If  $(\mathcal{C}, \mathcal{C})$  is an  $\mathcal{X}$ -mutation pair, that is, for any object  $A \in \mathcal{C}$  or  $B \in \mathcal{C}$  there exists an  $n$ -angle*

$$A \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} X_{n-2} \xrightarrow{\alpha_{n-1}} B \xrightarrow{\alpha_n} \Sigma A$$

where  $X_i \in \mathcal{X}$ ,  $\alpha_1$  is a left  $\mathcal{X}$ -approximation and  $\alpha_{n-1}$  is a right  $\mathcal{X}$ -approximation. Then the quotient  $\mathcal{C}/\mathcal{X}$  is an  $n$ -angulated category.

Before stating another corollary, we give some definitions. Let  $\mathcal{C}$  be an  $n$ -angulated category. An object  $I \in \mathcal{C}$  is called *injective* if for any morphism  $f : A \rightarrow B$  and any morphism  $g : A \rightarrow I$ , there exists a morphism  $h : B \rightarrow I$  such that  $g = hf$ . Denote by  $\mathcal{I}$  the subcategory of injectives. We say  $\mathcal{C}$  has enough injectives if for any object  $A \in \mathcal{C}$ , there exists an angle

$$A \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{n-2} \rightarrow B \rightarrow \Sigma A$$

with  $I_i \in \mathcal{I}$ . The notion of *having enough projectives* is defined dually. Denote by  $\mathcal{P}$  the subcategory of projectives. If  $\mathcal{C}$  has enough injectives, enough projectives and if injectives and projectives coincide, then we say  $\mathcal{C}$  is *Frobenius*.

The following result is a higher analogue of [1, Theorem 7.2].

**Corollary 3.11.** *Let  $\mathcal{C}$  be an  $n$ -angulated category.*

- (a) *If  $\mathcal{C}$  has enough injectives, then the quotient  $\mathcal{C}/\mathcal{I}$  is a right  $n$ -angulated category.*
- (b) *If  $\mathcal{C}$  has enough projectives, then the quotient  $\mathcal{C}/\mathcal{P}$  is a left  $n$ -angulated category.*
- (c) *If  $\mathcal{C}$  is Frobenius, then the quotient  $\mathcal{C}/\mathcal{I}$  is an  $n$ -angulated category.*

## REFERENCES

- [1] A. Beligiannis. Relative homological algebra and purity in triangulated categories. J. Algebra. 227, no.1, 268-361, 2000.
- [2] A. Beligiannis, N. Marmaridis. Left triangulated categories arising from contravariantly finite subcategories. Commun. Algebra. 22(12), 5021-5036, 1994.
- [3] P. A. Bergh, M. Thaule. The axioms for  $n$ -angulated categories. Algebr. Geom. Topol. 13(4), 2405-2428, 2013.
- [4] P. A. Bergh and M. Thaule. Higher  $n$ -angulations from local algebras. arXiv:1311.2089, 2013.
- [5] P. A. Bergh and M. Thaule. The Grothendieck group of an  $n$ -angulated category. J. Pure Appl. Algebra. 218(2), 354-366, 2014.
- [6] C. Geiss, B. Keller and S. Oppermann.  $n$ -angulated categories. J. Reine Angew. Math. 675, 101-120, 2013.
- [7] D. Happel. Triangulated categories in the representation theory of finite dimensional algebras. London Mathematical Society, LNS 119, Cambridge University Press, Cambridge, 1988.
- [8] G. Jasso.  $n$ -abelian and  $n$ -exact categories. arXiv:1405.7805v2, 2014.
- [9] Z. Lin.  $n$ -angulated quotient categories induced by mutation pairs. arXiv:1409.2716v1, 2014.

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